# QUANTUM INVARIANT, MODULAR FORM, AND LATTICE POINTS

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ABSTRACT. We study the Witten–Reshetikhin–Turaev SU(2) invariant for the Seifert manifold with 4-singular fibers. We define the Eichler integrals of the modular forms with half-integral weight, and we show that the invariant is rewritten as a sum of the Eichler integrals. Using a nearly modular property of the Eichler integral, we give an exact asymptotic expansion of the WRT invariant in $N \to \infty$. We reveal that the number of dominating terms, which is the number of the non-vanishing Eichler integrals in a limit $\tau \to N \in \mathbb{Z}$, is related to that of lattice points inside 4-dimensional simplex, and we discuss a relationship with the irreducible representations of the fundamental group.

## 1. INTRODUCTION

The SU(2) Witten invariant [31] for 3-manifold $\mathcal{M}$ is defined by

$$Z_k(\mathcal{M}) = \int \exp\Bigl(2\,\pi\,\mathrm{i}\,k\ \mathrm{CS}(A)\Bigr)\,\mathcal{D}A \tag{1.1}$$

where $k \in \mathbb{Z}$, and $\mathrm{CS}(A)$ is the Chern–Simons integral

$$\mathrm{CS}(A) = \frac{1}{8\,\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}\,A \wedge A \wedge A\right) \tag{1.2}$$

This invariant can be constructed rigorously using the quantum invariants of framed link as the Reshetikhin–Turaev invariant $\tau_N(\mathcal{M})$ [23], which is normalized as

$$Z_k(\mathcal{M}) = \frac{\tau_{k+2}(\mathcal{M})}{\tau_{k+2}\left(S^2 \times S^1\right)} \tag{1.3}$$

where

$$\tau_N\left(S^2 \times S^1\right) = \sqrt{\frac{N}{2}}\,\frac{1}{\sin(\pi/N)}$$

and we have

$$\tau_N\left(S^3\right) = 1$$

Studies on these quantum invariants have been extensively developed. Recently pointed out was a close relationship between the Witten–Reshetikhin–Turaev (WRT) invariant and modular form with half-integral weight. In Ref. 17, the WRT invariant for the Poincaré homology sphere $\Sigma(2,3,5)$ was identified with the Eichler integral of the modular form with weight $3/2$. This

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result was further developed in Ref. 7 where properties of the WRT invariant for the Brieskorn homology sphere $\Sigma(p, q, r)$ were investigated (see also Refs. 6, 8–10, in which clarified were the similar structures of the colored Jones polynomial for torus knots and links). One of benefits of the correspondence between the quantum invariant and the modular form is that we can obtain the exact asymptotic expansion from the modular property (see Refs. 4, 11, 14, 16, 24–27 for studies of asymptotic behavior of the SU(2) WRT invariant by different manner). We can also find that the number of the non-vanishing Eichler integrals in a limit $\tau \to N \in \mathbb{Z}$ coincides with that of the integral lattice points inside the 3-dimensional tetrahedron. We can then reinterpret the topological invariants such as the Casson invariant, the Reidemeister torsion, and the Chern–Simons invariant from the viewpoint of the modular form.

Purpose of this paper is to continue studies in Ref. 7, and to reveal a close connection between the WRT invariant for the Seifert manifold and the Eichler integral of the modular form. We especially study the Seifert manifold with 4-singular fibers $\Sigma(\vec{p}) = \Sigma(p_1, p_2, p_3, p_4)$ where $p_j$ are pairwise coprime integers. This manifold has a rational surgery description as in Fig. 1, and the fundamental group has the presentation

$$\pi_1\bigl(\Sigma(p_1,p_2,p_3,p_4)\bigr) = \left\langle x_1, x_2, x_3, x_4, h \;\middle|\; \begin{array}{c} h \text{ center} \\ x_k^{p_k} = h^{-q_k} \text{ for } k = 1,2,3,4 \\ x_1\, x_2\, x_3\, x_4 = 1 \end{array} \right\rangle \tag{1.4}$$

where $q_k \in \mathbb{Z}$ such that

$$P \sum_{j=1}^{4} \frac{q_j}{p_j} = 1 \tag{1.5}$$

Here and hereafter we use

$$P \equiv P(p_1, p_2, p_3, p_4) = \prod_{j=1}^{4} p_j$$

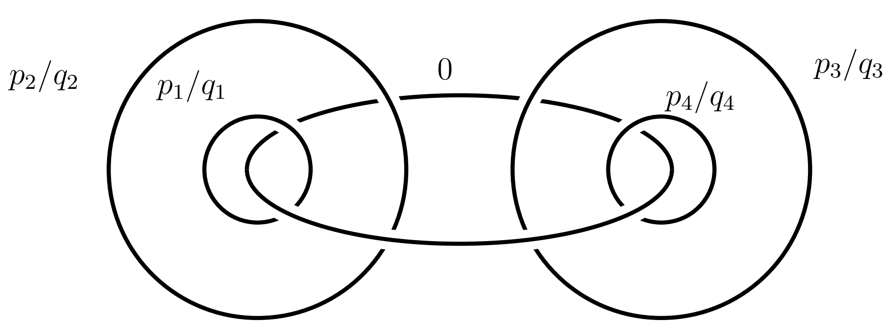


Figure 1: Rational surgery description of the Seifert manifold $\Sigma(p_1, p_2, p_3, p_4)$ with 4-singular fibers

This article is organized as follows. In Section 2 we briefly discuss construction of the WRT invariant for the Seifert manifold. We then prepare vector modular forms with half-integral weight, and we consider a nearly modular property of the Eichler integral. In Section 3 we prove that the WRT invariant for the Seifert manifold is written as a sum of the Eichler integrals of two types of the half-integral weight modular forms. Using a nearly modular property of the Eichler integrals, we obtain the exact asymptotic expansion of the WRT invariant in Section 4. We pay attention to dominating exponential factors of the WRT invariant, and show that the number of the non-vanishing terms is related to that of the irreducible representations of the fundamental group $\pi_1(\mathcal{M})$, and that they give the Chern–Simons invariant. We also compute the Ohtsuki invariant number-theoretically by use of the exact asymptotic expansion formula. In Section 5 we take some examples in detail, and compare numerically our asymptotic formula with the exact value of the quantum invariant. The last section is devoted to discussions.

## 2. Preliminaries

### 2.1 The Witten–Reshetikhin–Turaev Invariant of the Seifert Manifold

We compute the SU(2) WRT invariant for the Seifert homology manifold $\Sigma(p_1, p_2, p_3, p_4)$ with 4-singular fibers. In general the WRT invariant $\tau_N(\mathcal{M})$ for 3-manifold $\mathcal{M}$ can be constructed based on a surgery description on framed link. In our case, we need the colored Jones polynomial for a link depicted in Fig. 1. To construct the colored Jones polynomial for this link, we recall a fact that, when a link $\mathcal{L}$ is composed from three knots $\mathcal{K}_{0,1,2}$, the colored Jones polynomial for $\mathcal{L}$ is given by

$$J_{k_0,k_1,k_2}(\mathcal{K}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2) = \frac{J_{k_0,k_1}(\mathcal{K}_0 \cup \mathcal{K}_1)\, J_{k_0,k_2}(\mathcal{K}_0 \cup \mathcal{K}_2)}{J_{k_0}(\mathcal{K}_0)}$$

where $k_a$ denotes a color of knot $\mathcal{K}_a$. Using this property and an explicit form of the colored Jones invariant for the Hopf link, we see that the colored Jones polynomial for a link $\mathcal{L}$ in Fig. 1 is given by

$$J_{k_0,k_1,k_2,k_3,k_4}(\mathcal{L}) = \frac{1}{\sin(\pi/N)} \cdot \frac{\prod_{j=1}^{4} \sin\left(k_0\, k_j\, \pi/N\right)}{\big(\sin(k_0\, \pi/N)\big)^3} \tag{2.1}$$

Here $k_{j>0}$ is a color for knot which is to be $p_j/q_j$-surgery, and $k_0$ denotes a color for knot having a linking number 1 with other knots. We then apply a rational surgery formula presented in Ref. 11, and we finally obtain the following result. See Ref. 16 (also Ref. 7) for detail of computations.

**Proposition 1** ( [16]). *The WRT invariant $\tau_N(\mathcal{M})$ for the Seifert manifold $\mathcal{M} = \Sigma(p_1, p_2, p_3, p_4)$ is given by*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}(\frac{\phi}{4}-\frac{1}{2})} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N\left(\Sigma(p_1, p_2, p_3, p_4)\right) = \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{2\,P\,N}} \sum_{\substack{n=0 \\ N\nmid n}}^{2\,P\,N-1} \mathrm{e}^{-\frac{1}{2PN}n^2\pi\mathrm{i}} \frac{\prod_{j=1}^4 \left(\mathrm{e}^{\frac{n}{Np_j}\pi\mathrm{i}} - \mathrm{e}^{-\frac{n}{Np_j}\pi\mathrm{i}}\right)}{\left(\mathrm{e}^{\frac{n}{N}\pi\mathrm{i}} - \mathrm{e}^{-\frac{n}{N}\pi\mathrm{i}}\right)^2} \tag{2.2}$$

*where*

$$\phi \equiv \phi(p_1, p_2, p_3, p_4) = 3 - \frac{1}{P} + 12 \sum_{j=1}^{4} s\left(\frac{P}{p_j}, p_j\right) \tag{2.3}$$

Here we have used the Dedekind sum (see, *e.g.*, Ref. 22) defined by

$$s(b, a) = \sum_{k=1}^{a} \left(\left(\frac{k}{a}\right)\right) \cdot \left(\left(\frac{k\,b}{a}\right)\right) \tag{2.4}$$

with coprime integers $a \geq 1$ and $b$, and $((x))$ is the sawtooth function

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding $x$. It is known that the Dedekind sum is rewritten as the cotangent sum

$$s(b, a) = \frac{1}{4\,a} \sum_{k=1}^{a-1} \cot\left(\frac{k}{a}\,\pi\right) \cot\left(\frac{k\,b}{a}\,\pi\right)$$

and that the modular property of the logarithm of the Dedekind $\eta$-function $\eta(\tau)$ gives the reciprocity formula

$$s(b, a) + s(a, b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{a\,b}\right)$$

It is noted that

$$s(-b, a) = -s(b, a)$$

$$s(b, a) = s(c, a) \qquad \text{for } b\,c = 1 \pmod{a}$$

### 2.2 Modular Form

We fix $\vec{p} = (p_1, p_2, p_3, p_4)$ where $p_j$ are pairwise coprime positive integers. For a quadruple $\vec{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4) \in \mathbb{Z}^4$ satisfying $0 < \ell_j < p_j$, we define even periodic functions $\chi_{2P}^{\vec{\ell}}(n) =$

$\chi_{2P}^{(\ell_1,\ell_2,\ell_3,\ell_4)}(n)$ with modulus $2\,P$ as

$$\chi_{2P}^{\vec{\ell}}(n) = \begin{cases} -\prod_{j=1}^{4} \varepsilon_j & \text{if } n \equiv P \left(1 + \sum_{j=1}^{4} \varepsilon_j \frac{\ell_j}{p_j}\right) \mod 2\,P \\ 0 & \text{others} \end{cases} \tag{2.5}$$

where $\varepsilon_j \in \{1, -1\}$ for $\forall j$. There are $2^4 = 16$ non-zero $\chi_{2P}^{\vec{\ell}}(n)$ taking values $\pm 1$ for $0 < n < 2\,P$, and we have a mean value zero,

$$\sum_{n=0}^{2P-1} \chi_{2P}^{\vec{\ell}}(n) = 0 \tag{2.6}$$

We define an involution

$$\sigma_i(\vec{\ell}) = (\ell_1, \dots, p_i - \ell_i, \dots, p_4) \tag{2.7}$$

$$\begin{aligned} \sigma_{ij}(\vec{\ell}) &\equiv \sigma_i \circ \sigma_j(\vec{\ell}) \\ &= (\ell_1, \dots, p_i - \ell_i, \dots, p_j - \ell_j, \dots, p_4) \end{aligned} \tag{2.8}$$

for $1 \leq i \neq j \leq 4$. We see that the periodic function $\chi_{2P}^{\vec{\ell}}(n)$ is invariant under actions of $\sigma_{ij}$ and $\sigma_{12} \circ \sigma_{34}$;

$$\begin{aligned} \chi_{2P}^{\vec{\ell}}(n) &= \chi_{2P}^{\sigma_{ij}(\vec{\ell})}(n) \\ &= \chi_{2P}^{\sigma_{12}\circ\sigma_{34}(\vec{\ell})}(n) \end{aligned} \tag{2.9}$$

We note that

$$\chi_{2P}^{\vec{\ell}}(n + P) = -\chi_{2P}^{\sigma_i(\vec{\ell})}(n) \tag{2.10}$$

By means of the periodic functions $\chi_{2P}^{\vec{\ell}}(n)$, we define the $q$-series by

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \chi_{2P}^{\vec{\ell}}(n)\, q^{\frac{n^2}{4\,P}} \tag{2.11}$$

where as usual we have

$$q = \exp(2\,\pi\,\mathrm{i}\,\tau)$$

for $\tau$ in the upper half plane, $\tau \in \mathbb{H}$. Due to the symmetry of $\chi_{2P}^{\vec{\ell}}(n)$ under involutions (2.9), the number of the independent functions $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ is given by

$$D \equiv D(p_1, p_2, p_3, p_4) = \frac{1}{8} \prod_{j=1}^{4} (p_j - 1) \tag{2.12}$$

This set of functions $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ is a $D$-dimensional vector modular form with weight $1/2$; applying the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{e}^{-2\,\pi\,\mathrm{i}\,n\,t}\, f(t)\, \mathrm{d}t \tag{2.13}$$

we find under the $S$- and $T$-modular transformations that

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \sum_{\ell_1',\ell_2',\ell_3',\ell_4',} \mathbf{S}_{\vec{\ell}}^{\vec{\ell}'} \, \Phi_{\vec{p}}^{\vec{\ell}'}(-1/\tau) \tag{2.14}$$

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau+1) = \mathbf{T}^{\vec{\ell}} \, \Phi_{\vec{p}}^{\vec{\ell}}(\tau) \tag{2.15}$$

where a sum of quadruples $\vec{\ell}' = (\ell_1', \ell_2', \ell_3', \ell_4')$ runs over $D$-dimensional space, and explicit forms of the $\mathbf{S}$ and $\mathbf{T}$ matrices are respectively given by

$$\mathbf{S}_{\vec{\ell}}^{\vec{\ell}'} = \frac{16}{\sqrt{2\,P}} \, (-1)^{P\left(1+\sum_{j=1}^{4} \frac{\ell_j+\ell_j'}{p_j}\right) + P \sum_j \sum_{k\neq j} \frac{\ell_j \, \ell_k'}{p_j \, p_k}} \prod_{j=1}^{4} \sin\left( P \, \frac{\ell_j \, \ell_j'}{p_j^2} \, \pi \right) \tag{2.16}$$

$$\mathbf{T}^{\vec{\ell}} = \exp\left( \frac{P}{2} \left( 1 + \sum_{j=1}^{4} \frac{\ell_j}{p_j} \right)^2 \pi \, \mathrm{i} \right) \tag{2.17}$$

We further introduce other modular functions. We set

$$\Psi_P^{(a)}(\tau) = \frac{1}{2} \sum_{n\in\mathbb{Z}} n \, \psi_{2P}^{(a)}(n) \, q^{\frac{n^2}{4\,P}} \tag{2.18}$$

where $a \in \mathbb{Z}$ and $0 < a < P$, and $\psi_{2P}^{(a)}(n)$ is the odd periodic function

$$\psi_{2P}^{(a)}(n) = \begin{cases} \pm 1 & \text{for } n \equiv \pm a \mod 2\,P \\ 0 & \text{others} \end{cases} \tag{2.19}$$

By use of the Poisson summation formula (2.13) we see that the function $\Psi_P^{(a)}(\tau)$ is the ($P-1$)-dimensional vector modular form with weight $3/2$ satisfying

$$\Psi_P^{(a)}(\tau) = \left( \frac{\mathrm{i}}{\tau} \right)^{3/2} \sum_{b=1}^{P-1} \mathbf{M}_b^a \, \Psi_P^{(b)}(-1/\tau) \tag{2.20}$$

$$\Psi_P^{(a)}(\tau+1) = \exp\left( \frac{a^2}{2\,P} \, \pi \, \mathrm{i} \right) \, \Psi_P^{(a)}(\tau) \tag{2.21}$$

where $\mathbf{M}$ is a $(P-1) \times (P-1)$ matrix defined by

$$\mathbf{M}_b^a = \sqrt{\frac{2}{P}} \, \sin\left( \frac{a\,b}{P} \, \pi \right) \tag{2.22}$$

It should be remarked that we have

$$\Psi_{P=2}^{(1)}(\tau) = \big( \eta(\tau) \big)^3$$

and that the character of the affine Lie algebra $\widehat{su}(2)_{P-2}$ is given by (see *e.g.* Ref. 12)

$$\mathrm{ch}_a(\tau) = \frac{\Psi_P^{(a)}(\tau)}{\big(\eta(\tau)\big)^3}$$

### 2.3 Eichler Integral

Following Refs. 17, 32 we define the Eichler integrals of the half-integral weight modular forms $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ and $\Psi_P^{(a)}(\tau)$ as follows;

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau) = \sum_{n=0}^{\infty} n\,\chi_{2P}^{\vec{\ell}}(n)\,q^{\frac{n^2}{4P}} \tag{2.23}$$

$$\widetilde{\Psi}_P^{(a)}(\tau) = \sum_{n=0}^{\infty} \psi_{2P}^{(a)}(n)\,q^{\frac{n^2}{4P}} \tag{2.24}$$

which are defined for $\tau \in \mathbb{H}$. Limiting values of these Eichler integrals are given in the following Proposition.

**Proposition 2.** *Limiting values of the Eichler integrals $\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)$ and $\widetilde{\Psi}_P^{(a)}(\tau)$ at $\tau \to \frac{M}{N} \in \mathbb{Q}$ are respectively given by*

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(M/N) = -P\,N \sum_{k=1}^{2PN} \chi_{2P}^{\vec{\ell}}(k)\,\mathrm{e}^{\pi\mathrm{i}\frac{M}{N}\frac{k^2}{2P}}\,B_2\left(\frac{k}{2\,P\,N}\right) \tag{2.25}$$

$$\widetilde{\Psi}_P^{(a)}(M/N) = -\sum_{k=0}^{2PN} \psi_{2P}^{(a)}(k)\,\mathrm{e}^{\pi\mathrm{i}\frac{M}{N}\frac{k^2}{2P}}\,B_1\left(\frac{k}{2\,P\,N}\right) \tag{2.26}$$

*Here we assume $M$ and $N$ are relatively prime integers, and $N > 0$. We use $B_k(x)$ as the $k$-th Bernoulli polynomial*

$$\sum_{k=0}^{\infty} B_k(x)\,\frac{t^k}{k!} = \frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t - 1}$$

*and we have*

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

*Proof.* It is a standard result of applying the Mellin transformation, so we omit the proof. See Refs. 17, 32 (also Refs. 7, 9, 10). □

We note that a limiting value of the Eichler integrals at $\tau \to N \in \mathbb{Z}$ is a little bit simplified;

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(N) = -P \cdot C_{\vec{p}}(\vec{\ell}) \cdot \left(\mathbf{T}^{\vec{\ell}}\right)^N \tag{2.27}$$

$$\widetilde{\Psi}_P^{(a)}(N) = \left(1 - \frac{a}{P}\right) \mathrm{e}^{\frac{a^2}{2P}\pi \mathrm{i} N} \tag{2.28}$$

where the $\mathbf{T}$-matrix is defined in eq. (2.17), and we set

$$C_{\vec{p}}(\vec{\ell}) = \sum_{n=1}^{2P} \chi_{2P}^{\vec{\ell}}(n)\, B_2\left(\frac{n}{2\,P}\right) \tag{2.29}$$

**Proposition 3.** *The Eichler integrals have a nearly modular property. Especially asymptotic expansions in $N \to \infty$ are given by*

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(1/N) + \left(\frac{N}{\mathrm{i}}\right)^{3/2} \sum_{\vec{\ell}'} \mathbf{S}_{\vec{\ell}}^{\vec{\ell}'}\, \widetilde{\Phi}_{\vec{p}}^{\vec{\ell}'}(-N) \simeq \sum_{k=0}^{\infty} \frac{L(-2\,k-1, \chi_{2P}^{\vec{\ell}})}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \tag{2.30}$$

$$\widetilde{\Psi}_P^{(a)}(1/N) + \sqrt{\frac{N}{\mathrm{i}}} \sum_{b=1}^{P-1} \mathbf{M}_a^b\, \widetilde{\Psi}_P^{(b)}(-N) \simeq \sum_{k=0}^{\infty} \frac{L(-2\,k, \psi_{2P}^{(a)})}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \tag{2.31}$$

*Here $N \in \mathbb{Z}$, and a sum of quadruples $\vec{\ell}'$ runs over $D$-dimensional space. We mean that $L(s, \chi)$ is the Dirichlet $L$-series.*

*Proof.* We follow a proof given by Zagier for a case of weight $1/2$ [32] and of weight $3/2$ [17], and we shall give an outline of the proof below. See also Ref. 9 for a proof of eq. (2.31).

We introduce other Eichler integrals defined by

$$\widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(z) = -\sqrt{\frac{P\,\mathrm{i}}{2\,\pi^2}} \int_{z^*}^{\infty} \frac{\Phi_{\vec{p}}^{\vec{\ell}}(\tau)}{(\tau - z)^{3/2}}\, \mathrm{d}\tau \tag{2.32}$$

$$\widehat{\Psi}_P^{(a)}(z) = \frac{1}{\sqrt{2\,P\,\mathrm{i}}} \int_{z^*}^{\infty} \frac{\Psi_P^{(a)}(\tau)}{\sqrt{\tau - z}}\, \mathrm{d}\tau \tag{2.33}$$

both of which are defined for $z$ in the lower half plane, $z \in \mathbb{H}^-$, and $*$ denotes a complex conjugate. We see that the modular properties of $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ and $\Psi_P^{(a)}(\tau)$, especially the modular $S$-transformation (2.14) and (2.20), lead

$$\widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(z) + \left(\frac{1}{\mathrm{i}\,z}\right)^{3/2} \sum_{\vec{\ell}'} \mathbf{S}_{\vec{\ell}}^{\vec{\ell}'}\, \widehat{\Phi}_{\vec{p}}^{\vec{\ell}'}(-1/z) = r_{\Phi_{\vec{p}}}^{\vec{\ell}}(z;0) \tag{2.34}$$

$$\widehat{\Psi}_P^{(a)}(z) + \frac{1}{\sqrt{\mathrm{i}\,z}} \sum_{b=1}^{P-1} \mathbf{M}_b^a\, \widehat{\Psi}_P^{(b)}(-1/z) = r_{\Psi_P}^{(a)}(z;0) \tag{2.35}$$

where

$$r_{\Phi_{\vec{p}}}^{\vec{\ell}}(z;\alpha) = -\sqrt{\frac{P\,\mathrm{i}}{2\,\pi^2}}\int_\alpha^\infty \frac{\Phi_{\vec{p}}^{\vec{\ell}}(\tau)}{(\tau - z)^{3/2}}\,\mathrm{d}\tau$$

$$r_{\Psi_P}^{(a)}(z;\alpha) = \frac{1}{\sqrt{2\,P\,\mathrm{i}}}\int_\alpha^\infty \frac{\Psi_P^{(a)}(\tau)}{\sqrt{\tau - z}}\,\mathrm{d}\tau$$

with $\alpha \in \mathbb{Q}$. By substituting definitions of modular forms, we find that

$$\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(1/N) = \widehat{\Phi}_{\vec{p}}^{\vec{\ell}}(1/N)$$

$$\widetilde{\Psi}_P^{(a)}(1/N) = \widehat{\Psi}_P^{(a)}(1/N)$$

where LHSs are the limiting value from the upper plane $\mathbb{H}$ while RHSs are from the lower half plane $\mathbb{H}^-$. Taking asymptotic expansions of RHSs of both eqs. (2.34) and (2.35), we obtain eqs. (2.30) and (2.31). □

## 3. Quantum Invariant and Eichler Integrals

So far we have studied both the WRT invariant for the Seifert manifold and the modular forms with half-integral weight, and have computed limiting values of the Eichler integrals. One of our main theorems is that the WRT invariant for the Seifert manifold is written as a sum of the Eichler integrals. So the quadruple $\vec{p} = (p_1, p_2, p_3, p_4)$ used to define the modular form should be identified with the surgery data of the Seifert manifold.

**Theorem 4.** *The WRT invariant for the Seifert homology sphere $\Sigma(p_1, p_2, p_3, p_4)$, which is computed in eq.* (2.2)*, is rewritten in terms of a limiting value of the Eichler integrals.*

- $\sum_{j=1}^4 \frac{1}{p_j} < 1$*;*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi}{4}-\frac{1}{2}\right)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right)\tau_N\big(\Sigma(p_1,p_2,p_3,p_4)\big)$$
$$= \frac{1}{4\,P}\,\widetilde{\Phi}_{\vec{p}}^{(p_1-1,1,1,1)}(1/N) - \frac{1}{4\,P}\sum_{a=1}^{P-1} a\,\chi_{2P}^{(p_1-1,1,1,1)}(a)\,\widetilde{\Psi}_P^{(a)}(1/N) \quad (3.1)$$

- $\sum_{j=1}^4 \frac{1}{p_j} > 1$*;*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi}{4}-\frac{1}{2}\right)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right)\tau_N\big(\Sigma(p_1,p_2,p_3,p_4)\big)$$
$$= \frac{1}{4\,P}\,\widetilde{\Phi}_{\vec{p}}^{(p_1-1,1,1,1)}(1/N) - \frac{1}{4\,P}\sum_{a=1}^{P-1} a\,\chi_{2P}^{(p_1-1,1,1,1)}(a)\,\widetilde{\Psi}_P^{(a)}(1/N)$$
$$+ \frac{1}{2}\,\widetilde{\Psi}_P^{(2P-\sum_j \frac{P}{p_j})}(1/N) \quad (3.2)$$

To prove this theorem, we use the following formula;

**Lemma 5.** *For* $N, k \in \mathbb{Z}_{>0}$ *and* $0 \leq k \leq N-1$, *we have*

$$\sum_{n=1}^{N-1} \frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{N}(k+1)n}}{\left(1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}n}\right)^2} = \frac{1}{12} - \frac{N^2}{2}\, B_2\left(\frac{k}{N}\right) \tag{3.3}$$

*Proof.* We set $\omega = \exp\left(\frac{2\,\pi\,\mathrm{i}}{N}\right)$ for brevity. We recall a trivial identity

$$(1-x) \sum_{c=1}^{N-1} c\, x^c = \sum_{c=1}^{N} x^c - N\, x^N \tag{3.4}$$

Substituting $x = \omega^a$ for the above with $a \in \mathbb{Z}$ satisfying $0 < a < N$, we get

$$\frac{1}{1-\omega^a} = -\frac{1}{N} \sum_{c=1}^{N-1} c\, \omega^{ac} \tag{3.5}$$

where we have used $\sum_{c=1}^{N} \omega^{ac} = 0$. Differentiating eq. (3.4) w.r.t. $x$ and substituting $x = \omega^a$, we get

$$-2\, N\, \frac{\omega^a}{(1-\omega^a)^2} - N^2\, \frac{1}{1-\omega^a} = \sum_{c=1}^{N-1} c^2\, \omega^{ac} \tag{3.6}$$

We then find

$$\frac{\omega^a}{(1-\omega^a)^2} = \frac{1}{2\,N} \sum_{c=0}^{N-1} c\,(N-c)\, \omega^{ac}$$

Using this expression, we see that

$$\begin{aligned} \text{LHS of (3.3)} &= \frac{1}{2} \sum_{m=1}^{N-1} \sum_{c=0}^{N-1} c\, \left(1 - \frac{c}{N}\right)\, \omega^{m(k+c)} \\ &= \frac{1}{2} \sum_{c=0}^{N-1} c\, \left(1-\frac{c}{N}\right)\, \left(-1 + N\, \left(\delta_{k+c,0} + \delta_{k+c,N}\right)\right) \\ &= \frac{1}{12}\, \left(1 - N^2 + 6\, N\, k - 6\, k^2\right) \end{aligned}$$

which proves eq. (3.3). See Ref. 1 for elegant treatments of several identities concerning the $N$-th root of unity. □

*Proof of Theorem 4.* We first consider a case $\sum_j \frac{1}{p_j} < 1$. We find that there is a generating function for the periodic function $\chi_{2P}^{(1,1,1,1)}(n)$;

$$-z^P \prod_{j=1}^{4} \left(z^{P/p_j} - z^{-P/p_j}\right) = \sum_{n=0}^{2P-1} \chi_{2P}^{(1,1,1,1)}(n)\, z^n \tag{3.7}$$

Using above identity, we get

$$
\begin{aligned}
&\text{LHS of eq. (3.1)}\\
&= \frac{-\mathrm{e}^{\frac{\pi \mathrm{i}}{4}}}{2\sqrt{2\,P\,N}} \sum_{j=0}^{2P-1}\sum_{m=1}^{N-1}\sum_{k=0}^{2P-1} \mathrm{e}^{-\frac{(Nj+m-k)^2}{2PN}\pi\mathrm{i}+\frac{k^2}{2PN}\pi\mathrm{i}}\, \chi_{2P}^{(1,1,1,1)}(k)\, \frac{\mathrm{e}^{(j+\frac{m}{N})\pi\mathrm{i}}}{\left(1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}m}\right)^2}\\
&= \frac{-1}{2\,N} \sum_{k=0}^{2P-1}\sum_{j=0}^{N-1} \chi_{2P}^{(1,1,1,1)}(k)\, \mathrm{e}^{\frac{\pi\mathrm{i}}{2PN}\left(k-2P(j+\frac{1}{2})\right)^2} \sum_{m=1}^{N-1} \frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}(j+1)m}}{\left(1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}m}\right)^2}\\
&= -\frac{1}{2\,N}\sum_{n=0}^{2PN-1} \chi_{2P}^{(1,1,1,1)}(n)\, \mathrm{e}^{\frac{\pi\mathrm{i}}{2PN}(n-P)^2}\, \left\{ \frac{1}{12} - \frac{N^2}{2}\, B_2\left(\frac{1}{N}\left\lfloor \frac{n}{2\,P}\right\rfloor\right)\right\}
\end{aligned}
$$

Here in the second equality, we have used the Gauss sum reciprocity formula (see *e.g.* Ref. 11)

$$
\sum_{n \mod N} \mathrm{e}^{\frac{\pi\mathrm{i}}{N}Mn^2+2\pi\mathrm{i}kn} = \sqrt{\left|\frac{N}{M}\right|}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\,\mathrm{sign}(NM)} \sum_{n \mod M} \mathrm{e}^{-\frac{\pi\mathrm{i}}{M}N(n+k)^2} \tag{3.8}
$$

where $N, M \in \mathbb{Z}$ with $N\,k \in \mathbb{Z}$ and $N\,M$ being even. In the third equality we have applied eq. (3.3), and have used properties of the Bernoulli polynomials,

$$
B_k(1-x) = (-1)^k\, B_k(x)
$$

$$
B_k(x+1) - B_k(x) = k\, x^{k-1}
$$

We then obtain

$$
\begin{aligned}
\text{LHS of eq. (3.1)} = \frac{N}{4}\sum_{n=0}^{2PN-1} &\chi_{2P}^{(1,1,1,1)}(n+P)\,\mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}}\\
\times\Bigg\{ B_2\left(\frac{n}{2\,P\,N}\right) &- \frac{2}{N}\, B_1\left(\frac{n}{2\,P\,N}\right)\, B_1\left(\frac{n+P}{2\,P} - \left\lfloor\frac{n+P}{2\,P}\right\rfloor\right)\\
&+ \frac{1}{N^2}\, B_2\left(\frac{n+P}{2\,P} - \left\lfloor\frac{n+P}{2\,P}\right\rfloor\right) - \frac{1}{12\,N^2}\Bigg\}
\end{aligned} \tag{3.9}
$$

Using eq. (2.10), one sees that the first term gives $\frac{1}{4\,P}\,\widetilde{\Phi}_{\vec{p}}^{\sigma_i(1,1,1,1)}(1/N)$, and that the second term is written in terms of $\widetilde{\Psi}_P^{(a)}(1/N)$ as the function $B_1\left(x+\frac{1}{2}-\left\lfloor x+\frac{1}{2}\right\rfloor\right)$ is an odd periodic sawtooth function satisfying

$$
\chi_{2P}^{\vec{\ell}}(n)\, B_1\left(\frac{n+P}{2\,P} - \left\lfloor\frac{n+P}{2\,P}\right\rfloor\right) = \frac{1}{2\,P}\sum_{a=1}^{P} a\,\chi_{2P}^{\vec{\ell}}(a)\,\psi_{2P}^{(a)}(n) \tag{3.10}
$$

To obtain eq. (3.1), we need to prove that the remaining terms vanish. To see this, we have for a case of the fourth constant term in eq. (3.9)

$$\begin{aligned}
&\sum_{n=0}^{2PN-1} \chi_{2P}^{(1,1,1,1)}(n+P)\,\mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}} \\
&= -\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2\,P\,N}} \sum_{n=0}^{2PN-1}\sum_{k=0}^{2PN-1} \chi_{2P}^{(p_1-1,1,1,1)}(n)\mathrm{e}^{-\frac{k^2}{2PN}\pi\mathrm{i}+\frac{kn}{PN}\pi\mathrm{i}} \\
&= -\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2\,P\,N}} \sum_{j=0}^{2P-1}\sum_{k=0}^{2PN-1} \chi_{2P}^{(p_1-1,1,1,1)}(j)\,\left(\sum_{m=0}^{N-1}\mathrm{e}^{\frac{2km}{N}\pi\mathrm{i}}\right)\,\mathrm{e}^{-\frac{k^2}{2PN}\pi\mathrm{i}+\frac{kj}{PN}\pi\mathrm{i}}
\end{aligned}$$

where in the first equality we have used eq. (2.10) and eq. (3.8). In the second equality we have simply set $n = 2\,P\,m + j$. As the sum in the parenthesis in the last expression vanishes, we can conclude that the last term in eq. (3.9) vanishes. When we recall the Fourier expansion of the periodic Bernoulli polynomials (cf. Ref. 1)

$$B_k\,(x - \lfloor x \rfloor) = -\frac{k!}{(2\,\pi\,\mathrm{i})^k} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}nx}}{n^k}$$

for $k \in \mathbb{Z}_{>0}$, we can see that the third term in eq. (3.9) also vanishes in the same manner. This completes the proof of Theorem 4 for a case of $\sum_j \frac{1}{p_j} < 1$.

In a case of $\sum_j \frac{1}{p_j} > 1$, the generating function (3.7) is replaced with

$$\begin{aligned}
&- z^P \prod_{j=1}^{4}\left(z^{P/p_j} - z^{-P/p_j}\right) + z^P\,\left(z^P - z^{-P}\right)\,\left(z^{P\sum_j \frac{1}{p_j} - P} - z^{-P\sum_j \frac{1}{p_j} + P}\right) \\
&\qquad = \sum_{n=0}^{2P-1} \chi_{2P}^{(1,1,1,1)}(n)\,z^n \qquad (3.11)
\end{aligned}$$

*i.e.* we have an additional term coming from the second term in LHS of eq. (3.11). Then we have another contribution to the WRT invariant besides eq. (3.9);

$$\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{2\sqrt{2\,P\,N}} \sum_{\substack{n=0\\ N\nmid n}}^{2\,P\,N-1} \mathrm{e}^{-\frac{n^2}{2PN}\pi\mathrm{i}}\, \frac{\mathrm{e}^{\frac{n}{N}\pi\mathrm{i}\left(\sum_j \frac{1}{p_j}-1\right)} - \mathrm{e}^{-\frac{n}{N}\pi\mathrm{i}\left(\sum_j \frac{1}{p_j}-1\right)}}{\mathrm{e}^{\frac{n}{N}\pi\mathrm{i}} - \mathrm{e}^{-\frac{n}{N}\pi\mathrm{i}}} \qquad (3.12)$$

This term can also be rewritten in terms of the Eichler integral. To see this fact, we compute as follows [7, 17];

$$\begin{aligned}\widetilde{\Psi}_P^{(2P-P\sum_j \frac{1}{p_j})}(1/N) &= \lim_{t\searrow 0}\sum_{n=0}^{\infty}\psi_{2P}^{(2P-P\sum_j \frac{1}{p_j})}(n)\,\mathrm{e}^{\frac{n^2}{2PN}\pi\mathrm{i}-nt}\\ &= \lim_{t\searrow 0}\sum_{n=0}^{\infty}\sum_{k=0}^{2PN-1}\psi_{2P}^{(2P-P\sum_j \frac{1}{p_j})}(n)\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2\,P\,N}}\,\mathrm{e}^{-\frac{k^2}{2PN}\pi\mathrm{i}+\frac{kn}{PN}\pi\mathrm{i}-nt}\\ &= \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2\,P\,N}}\sum_{\substack{k=0\\ N\nmid k}}^{2PN-1}\mathrm{e}^{-\frac{k^2}{2PN}\pi\mathrm{i}}\,\frac{\mathrm{e}^{\frac{k}{N}\pi\mathrm{i}\left(\sum_j \frac{1}{p_j}-1\right)}-\mathrm{e}^{-\frac{k}{N}\pi\mathrm{i}\left(\sum_j \frac{1}{p_j}-1\right)}}{\mathrm{e}^{\frac{k}{N}\pi\mathrm{i}}-\mathrm{e}^{-\frac{k}{N}\pi\mathrm{i}}}\end{aligned}$$

In the last equality we have used a generating function of the periodic function $\psi_{2P}^{(a)}(n)$, and also used a fact that the sum for $N|k$ vanishes. This completes the proof of the theorem. □

## 4. Asymptotic Expansion of the WRT Invariant

We have seen that the WRT invariant $\tau_N(\mathcal{M})$ for the Seifert manifold $\mathcal{M} = \Sigma(p_1, p_2, p_3, p_4)$ is written as a sum of the Eichler integrals of the modular forms with half-integral weight. As we have already found a nearly modular property of these Eichler integrals, it is straightforward to obtain the following theorem.

**Theorem 6.** *Asymptotic expansion of the WRT invariant $\tau_N(\mathcal{M})$ for the Seifert homology sphere $\mathcal{M} = \Sigma(p_1, p_2, p_3, p_4)$ in $N \to \infty$ is given as follows.*

- $\sum_{j=1}^{4}\frac{1}{p_j} < 1$;

$$\begin{aligned}&\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi}{4}-\frac{1}{2}\right)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right)\tau_N\left(\Sigma(p_1,p_2,p_3,p_4)\right)\\ &\qquad\simeq -\frac{1}{4\,P}\left(\frac{N}{\mathrm{i}}\right)^{3/2}\sum_{\vec{\ell}}\mathbf{S}_{p_1-1,1,1,1}^{\vec{\ell}}\,\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(-N)\\ &\quad+\sqrt{\frac{N}{\mathrm{i}}}\sum_{b=1}^{P-1}\left(\sum_{a=1}^{P-1}a\,\chi_{2P}^{(p_1-1,1,1,1)}(a)\,\sin\left(\frac{a\,b}{P}\,\pi\right)\right)\frac{P-b}{\sqrt{8\,P^5}}\,\mathrm{e}^{-\frac{b^2}{2\,P}\,\pi\,\mathrm{i}\,N}\\ &\qquad+\sum_{k=0}^{\infty}\frac{T_<(k)}{k!}\left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k\end{aligned} \tag{4.1}$$

*where the $T$-series $T_<(k)$ is defined by*

$$T_<(k) = \frac{1}{4\,P} \left( L(-2\,k-1, \chi_{2P}^{(p_1-1,1,1,1)}) - \sum_{a=1}^{P-1} a\,\chi_{2P}^{(p_1-1,1,1,1)}(a)\, L(-2\,k, \psi_{2P}^{(a)}) \right) \tag{4.2}$$

$$= (2\,P)^{2k} \sum_{n=-P}^{P} \chi_{2P}^{(p_1-1,1,1,1)}(n) \left( -\frac{1}{4\,(k+1)}\, B_{2k+2}\left(\frac{n}{2\,P}\right) + \frac{n}{4\,P\,(2\,k+1)}\, B_{2k+1}\left(\frac{n}{2\,P}\right) \right)$$

- $\sum_{j=1}^{4} \frac{1}{p_j} > 1$;

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\left(\frac{\phi}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N\big(\Sigma(p_1,p_2,p_3,p_4)\big)$$

$$\simeq -\frac{1}{4\,P} \left(\frac{N}{\mathrm{i}}\right)^{3/2} \sum_{\vec{\ell}'} \mathbf{S}_{p_1-1,1,1,1}^{\vec{\ell}'}\, \widetilde{\Phi}_{\vec{p}}^{\vec{\ell}'}(-N)$$

$$+ \sqrt{\frac{N}{\mathrm{i}}} \sum_{b=1}^{P-1} \left( \sum_{a=1}^{P-1} a\,\chi_{2P}^{(p_1-1,1,1,1)}(a)\, \sin\left(\frac{a\,b}{P}\,\pi\right)\right) \frac{P-b}{\sqrt{8\,P^5}}\, \mathrm{e}^{-\frac{b^2}{2\,P}\,\pi\,\mathrm{i}\,N}$$

$$+ \sqrt{\frac{N}{\mathrm{i}}} \sum_{b=1}^{P-1} \sin\left( \sum_j \frac{1}{p_j}\, b\,\pi \right) \cdot \frac{P-b}{\sqrt{2\,P^3}}\, \mathrm{e}^{-\frac{b^2}{2P}\,\pi\,\mathrm{i}\,N}$$

$$+ \sum_{k=0}^{\infty} \frac{T_>(k)}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k \tag{4.3}$$

*where the $T$-series $T_>(k)$ is given by*

$$T_>(k) = \frac{1}{4\,P} \left( L(-2\,k-1, \chi_{2P}^{(p_1-1,1,1,1)}) - \sum_{a=1}^{P-1} a\,\chi_{2P}^{(p_1-1,1,1,1)}(a)\, L(-2\,k, \psi_{2P}^{(a)}) \right)$$

$$+ \frac{1}{2}\, L(-2\,k, \psi_{2P}^{(2P-P\sum_j \frac{1}{p_j})}) \tag{4.4}$$

$$= (2\,P)^{2k} \sum_{n=-P}^{P} \chi_{2P}^{(p_1-1,1,1,1)}(n) \left( -\frac{1}{4\,(k+1)}\, B_{2k+2}\left(\frac{n}{2\,P}\right) + \frac{n}{4\,P\,(2\,k+1)}\, B_{2k+1}\left(\frac{n}{2\,P}\right) \right)$$

$$+ \frac{(2\,P)^{2k}}{2\,k+1}\, B_{2k+1}\left( \frac{1}{2} \sum_j \frac{1}{p_j} \right)$$

### 4.1 Lattice Points and Non-Vanishing Eichler Integral

We consider dominating terms of the WRT invariant (4.1) and (4.3) in $N \to \infty$ in detail. We shall reveal a close connection with the irreducible representation of the fundamental group $\pi_1(\mathcal{M})$ and the Chern–Simons invariant $\mathrm{CS}(\mathcal{M})$ for the Seifert manifold $\mathcal{M} = \Sigma(p_1,p_2,p_3,p_4)$.

Theorem 6 indicates that the WRT invariant has exponentially divergent terms in $N \to \infty$. Recalling an explicit form of the $\mathbf{S}$-matrix (2.16) we get the following formula.

**Corollary 7.** *The Witten invariant for the Seifert homology sphere $\Sigma(p_1, p_2, p_3, p_4)$ behaves in $N \to \infty$ as*

$$Z_{N-2}\big(\Sigma(p_1,p_2,p_3,p_4)\big) \sim N\, \mathrm{e}^{\frac{3}{4}\pi\mathrm{i} - \frac{\phi}{2N}\pi\mathrm{i}} \sum_{\vec{\ell}} (-1)^{1+P\sum_k \frac{1}{p_k} + P\sum_{j\neq k}\frac{\ell_k}{p_j p_k}} \times \frac{2}{\sqrt{P}} \left( \prod_{k=1}^{4} \sin\left(P\, \frac{\ell_k}{p_k^2}\,\pi\right)\right) C_{\vec{p}}(\vec{\ell})\, \mathrm{e}^{-\frac{P}{2}\left(1+\sum_k \frac{\ell_k}{p_k}\right)^2 \pi\mathrm{i} N} \tag{4.5}$$

*where the sum of $\vec{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4)$ runs over $D$-dimensional space, and $C_{\vec{p}}(\vec{\ell})$ is defined in eq.* (2.29).

We note that, when we collect all the exponentially divergent terms in eq. (4.1) and (4.3), we have

$$Z_{N-2}\big(\Sigma(\vec{p})\big) \sim N \cdot Z^{(0)}_{N-2}\big(\Sigma(\vec{p})\big) + Z^{(1)}_{N-2}\big(\Sigma(\vec{p})\big) \tag{4.6}$$

Here the leading term $N \cdot Z^{(0)}_{N-2}(\Sigma(\vec{p}))$ comes from a limiting value (2.27) of the Eichler integral $\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(-N)$,

$$Z^{(0)}_{N-2}\big(\Sigma(p_1,p_2,p_3,p_4)\big) = \mathrm{e}^{\frac{3}{4}\pi\mathrm{i} - \frac{\phi}{2N}\pi\mathrm{i}} \sum_{\vec{\ell}} (-1)^{1+\sum_k \frac{P}{p_k} + P\sum_{j\neq k}\frac{\ell_k}{p_j p_k}} \times \frac{2}{\sqrt{P}} \left( \prod_{k} \sin\left(P\, \frac{\ell_k}{p_k^2}\,\pi\right)\right) C_{\vec{p}}(\ell)\, \mathrm{e}^{-\frac{P}{2}\left(1+\sum_k \frac{\ell_k}{p_k}\right)^2 \pi\mathrm{i} N} \tag{4.7}$$

The term $Z^{(1)}_{N-2}$ denotes the next leading term, which follows from the limiting value of the Eichler integral $\widetilde{\Psi}^{(a)}_{P}(-N)$;

- for $\sum_j \frac{1}{p_j} > 1$

$$Z^{(1)}_{N-2}\big(\Sigma(p_1,p_2,p_3,p_4)\big) = \mathrm{e}^{-\frac{3}{4}\pi\mathrm{i} - \frac{\phi}{2N}\pi\mathrm{i}} \sum_{b=1}^{P-1} \frac{P-b}{2\sqrt{P^3}} \left( \sin\left(\sum_j \frac{b}{p_j}\pi\right) - 4\sum_j \frac{1}{p_j}\,\cos\left(\frac{b}{p_j}\,\pi\right) \prod_{k\neq j} \sin\left(\frac{b}{p_k}\,\pi\right)\right) \mathrm{e}^{-\frac{b^2}{2P}\pi\,\mathrm{i}\,N} \tag{4.8}$$

- for $\sum_j \frac{1}{p_j} < 1$

$$Z^{(1)}_{N-2}\big(\Sigma(p_1,p_2,p_3,p_4)\big) = \mathrm{e}^{-\frac{3}{4}\pi\mathrm{i} - \frac{\phi}{2N}\pi\mathrm{i}} \sum_{b=1}^{P-1} \frac{2\,(b-P)}{\sqrt{P^3}} \left( \sum_j \frac{1}{p_j}\,\cos\left(\frac{b}{p_j}\,\pi\right) \prod_{k\neq j} \sin\left(\frac{b}{p_k}\,\pi\right)\right) \mathrm{e}^{-\frac{b^2}{2P}\pi\,\mathrm{i}\,N} \tag{4.9}$$

The sum of quadruples $\vec{\ell}$ in the leading term $Z^{(0)}_{N-2}\bigl(\Sigma(\vec{p})\bigr)$ runs over $D$-dimensional space, but as in the case of the Brieskorn homology sphere $\Sigma(p_1,p_2,p_3)$ with 3-singular fibers [7], the function $C_{\vec{p}}(\vec{\ell})$ may vanish for some quadruples $\vec{\ell}$.

**Proposition 8.** *Let $\gamma(p_1,p_2,p_3,p_4)$ be the number of the Eichler integrals $\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(\tau)$ which do not vanish in a limit $\tau \to N \in \mathbb{Z}$. Then $D-\gamma(p_1,p_2,p_3,p_4)$ coincides with the number of the integral lattice points $\vec{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in\mathbb{Z}^4$ satisfying*

$$0 < \frac{\ell_1}{p_1}+\frac{\ell_2}{p_2}+\frac{\ell_3}{p_3}+\frac{\ell_4}{p_4} < 1 \tag{4.10}$$

*where $0<\ell_j<p_j$. Namely the number of lattice points inside the integral simplex whose vertices are $(p_1,0,0,0)$, $(0,p_2,0,0)$, $(0,0,p_3,0)$, $(0,0,0,p_4)$, and the origin $(0,0,0,0)$.*

*Proof.* Eq. (2.27) indicates that the limiting value $\widetilde{\Phi}^{\vec{\ell}}_{\vec{p}}(N)$ of the Eichler integral does not vanish if and only if

$$C_{\vec{p}}(\vec{\ell}) \neq 0 \tag{4.11}$$

We consider a condition for quadruples $\vec{\ell}$. As we have $0<\frac{\ell_j}{p_j}<1$ by definition, we have $0<\sum_{j=1}^4 \frac{\ell_j}{p_j}<4$, and a pairwise coprime condition of $p_j$ indicates $\sum_j \frac{\ell_j}{p_j}\notin\mathbb{Z}$.

We first assume $0<\sum_j\frac{\ell_j}{p_j}<1$. Here we use $\{a,b,c,d\}=\{1,2,3,4\}$. From $0<\frac{\ell_j}{p_j}<1$ we see that $0<1+\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}+\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<2$. If we have $-1<1+\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}-\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<0$, we have a contradiction $\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}<0$ by summing with the assumption $0<\sum_j\frac{\ell_j}{p_j}<1$. Then we see that $0<1+\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}-\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<2$ for arbitrary setting of $\{a,b,c,d\}$. Then collecting non-zero 16 terms of $\chi^{\vec{\ell}}_{2P}(n)$, we can check by a direct computation that

$$\begin{aligned} C_{\vec{p}}(\vec{\ell}) &= -\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4=\pm1}\left(\prod_{j=1}^4\varepsilon_j\right) B_2\left(\frac{1}{2}\left(1+\sum_{j=1}^4\varepsilon_j\frac{\ell_j}{p_j}\right)\right) \\ &= 0 \end{aligned}$$

We next consider a case of $1<\sum_j\frac{\ell_j}{p_j}<3$. If we have $-1<1+\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}-\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<0$ and $-1<1+\frac{\ell_a}{p_a}-\frac{\ell_b}{p_b}+\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<0$ for some $\{a,b,c,d\}$, we get $1+\frac{\ell_a}{p_a}-\frac{\ell_d}{p_d}<0$ which contradicts with $0<\frac{\ell_j}{p_j}<1$. Thus, as inequalities for $1+\sum_j\varepsilon_j\frac{\ell_j}{p_j}$ with $\#\{\varepsilon_j=1\}=2$, we have two possibilities;

- $0<1+\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}-\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<2$ for every setting of $\{a,b,c,d\}$,
- for a unique setting of $\{a,b,c,d\}$ among 6 we have $-1<1+\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}-\frac{\ell_c}{p_c}-\frac{\ell_d}{p_d}<0$ and $2<1-\frac{\ell_a}{p_a}-\frac{\ell_b}{p_b}+\frac{\ell_c}{p_c}+\frac{\ell_d}{p_d}<3$

In the latter case, we see easily $C_{\vec{p}}(\vec{\ell}) = 0$ as the condition coincides with $0 < \sigma_{cd}\left(\sum_j \frac{\ell_j}{p_j}\right) < 1$. For a case of the former, we still need to classify a condition for $1 + \frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} - \frac{\ell_d}{p_d}$ into $0 < 1 + \frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} - \frac{\ell_d}{p_d} < 2$ or $2 < 1 + \frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} - \frac{\ell_d}{p_d} < 3$.

- When $0 < 1 + \frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} - \frac{\ell_d}{p_d} < 2$ for any setting of $\{a, b, c, d\}$, we have

$$C_{\vec{p}}(\vec{\ell}) = 2\left(\sum_j \frac{\ell_j}{p_j} - 1\right)$$

- When, among $1+\sum_j \varepsilon_j \frac{\ell_j}{p_j}$ with $\#\{\varepsilon_j = -1\} = 1$, we have, say $2 < 1-\frac{\ell_a}{p_a}+\frac{\ell_b}{p_b}+\frac{\ell_c}{p_c}+\frac{\ell_d}{p_d} < 3$ for a unique setting of $\{a, b, c, d\}$ and others in $[0, 2]$, we see

$$C_{\vec{p}}(\vec{\ell}) = 4\,\frac{\ell_a}{p_a}$$

- When two combinations of $1 + \sum_j \varepsilon_j \frac{\ell_j}{p_j}$ with $\#\{\varepsilon_j = -1\} = 1$ are in $[2, 3]$, say $2 < 1 - \frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} + \frac{\ell_d}{p_d} < 3$, $2 < 1 + \frac{\ell_a}{p_a} - \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} + \frac{\ell_d}{p_d} < 3$, and others in $[0, 2]$, we have

$$C_{\vec{p}}(\vec{\ell}) = 2\left(\frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} - \frac{\ell_c}{p_c} - \frac{\ell_d}{p_d}\right)$$

- If three combinations are in $[2, 3]$, say $0 < 1 + \frac{\ell_a}{p_a} + \frac{\ell_b}{p_b} + \frac{\ell_c}{p_c} - \frac{\ell_d}{p_d} < 2$ for a unique setting of $\{a, b, c, d\}$, we get

$$C_{\vec{p}}(\vec{\ell}) = 4\left(1 - \frac{\ell_d}{p_d}\right)$$

As a condition $3 < \sum_j \frac{\ell_j}{p_j} < 4$ means $0 < \sigma_{12} \circ \sigma_{34}\left(\sum_j \frac{\ell_j}{p_j}\right) < 1$, we get $C_{\vec{p}}(\vec{\ell}) = 0$ for this case.

Combining these observations, we find that $C_{\vec{p}}(\vec{\ell}) = 0$ if $0 < \sum_j \frac{\ell_j}{p_j} < 1$ or $0 < \sigma_{ab}\left(\sum_j \frac{\ell_j}{p_j}\right) < 1$ or $3 < \sum_j \frac{\ell_j}{p_j} < 4$. Recalling the invariance of $\chi_{2P}^{\vec{\ell}}(n)$ under the involution $\sigma_{ab}$, we can conclude that $D - \gamma(p_1, p_2, p_3, p_4)$ coincides with the number of integral lattice points satisfying eq. (4.10). □

Computation of the number of the integral lattice points inside polytopes is an old but difficult problem. We have simple but beautiful Pick's formula in the case of the 2-dimensional integral polygons. In the three-dimensional case, the number of the lattice points inside the integral tetrahedron was computed by Mordell using the Dedekind sum [18]. In our case of the 4-dimensional simplex, a formula was also given by Mordell [18], and Prop. 8 proves the following theorem.

**Theorem 9.** *The number of the non-vanishing Eichler integral* $\widetilde{\Phi}_{\vec{p}}^{\vec{\ell}}(\tau)$ *at* $\tau \to N \in \mathbb{Z}$ *is given by*

$$\gamma(p_1,p_2,p_3,p_4) = -\frac{3}{8} + \frac{P}{12} - \frac{P}{24} \sum_{j=1}^{4} \frac{1+p_j}{{p_j}^2} - \frac{1}{24\,P} \left(1 - \sum_{j=1}^{4} p_j\right) + \frac{P}{24} \sum_{j \neq k}^{4} \frac{1}{{p_j}^2\, p_k} + \frac{1}{2} \sum_{j=1}^{4} s\left(\frac{P}{p_j}, p_j\right) - \frac{1}{2} \sum_{j \neq k}^{4} s\left(\frac{P}{p_j\, p_k}, p_j\right) \tag{4.12}$$

*where* $s(b,a)$ *is the Dedekind sum* (2.4).

As the number $\gamma(\vec{p})$ of the non-vanishing Eichler integrals corresponds to that of the dominating terms (4.5) of the WRT invariant, and it could be related to the Casson invariant. The Casson invariant of the Seifert manifold $\Sigma(p_1,p_2,\dots,p_M)$ with $M$-singular fibers is defined naïvely as the number of the nontrivial SU(2) representations of $\pi_1\bigl(\Sigma(p_1,\dots,p_M)\bigr)$ [30] (see also Refs. 2, 29). It is known to be written explicitly as [5, 20]

$$\lambda_C\bigl(\Sigma(p_1,p_2,\dots,p_M)\bigr) = -\frac{1}{8} + \frac{1}{24\,P_M} \left(1 + \sum_{j=1}^{M} \left(\frac{P_M}{p_j}\right)^2 - (M-2)\, P_M^2\right) - \frac{1}{2} \sum_{j=1}^{M} s\left(\frac{P_M}{p_j}, p_j\right) \tag{4.13}$$

where we have used $P_M = \prod_{j=1}^{M} p_j$. Using this result we obtain the following expression.

**Corollary 10.** *We have*

$$\gamma(p_1,p_2,p_3,p_4) = \lambda_C\bigl(\Sigma(p_1,p_2,p_3)\bigr) + \lambda_C\bigl(\Sigma(p_1,p_2,p_4)\bigr) + \lambda_C\bigl(\Sigma(p_1,p_3,p_4)\bigr) + \lambda_C\bigl(\Sigma(p_2,p_3,p_4)\bigr) - \lambda_C\bigl(\Sigma(p_1,p_2,p_3,p_4)\bigr) \tag{4.14}$$

*where* $\lambda_C(\mathcal{M})$ *denotes the Casson invariant for 3-manifold* $\mathcal{M}$.

We should note that, in terms of the function $\phi(p_1,p_2,p_3,p_4)$ defined in eq. (2.3), we have [16]

$$-24 \cdot \lambda_C\bigl(\Sigma(p_1,p_2,p_3,p_4)\bigr) = \phi + P\left(2 - \sum_{j=1}^{4} \frac{1}{{p_j}^2}\right) \tag{4.15}$$

The asymptotic behavior of the SU(2) WRT invariant for 3-manifold $\mathcal{M}$ in $k \to \infty$ is expected to be [4, 31]

$$Z_k(\mathcal{M}) \sim \frac{1}{2}\, \mathrm{e}^{-\frac{3}{4}\pi\,\mathrm{i}} \sum_{\alpha} \sqrt{T_\alpha(\mathcal{M})}\, \mathrm{e}^{-2\pi\,\mathrm{i}\, I_\alpha/4}\, \mathrm{e}^{2\pi\mathrm{i}(k+2)\,\mathrm{CS}(A)} \tag{4.16}$$

where $T_\alpha(\mathcal{M})$ and $I_\alpha$ are respectively the Reidemeister–Ray–Singer torsion and the spectral flow, and the sum of $\alpha$ denotes a flat connection.

It is known [7, 24] that, in the case of the Brieskorn homology sphere $\Sigma(p_1,p_2,p_3)$, the exact asymptotic expansion of the WRT invariant has a form of eq. (4.16) by identifying SU(2) representation of the fundamental group $\pi_1(\mathcal{M})$ with flat connections on $\mathcal{M}$, and that the Casson invariant $\lambda_C\big(\Sigma(p_1,p_2,p_3)\big)$ is equal to a minus one-half of the number of non-zero terms in eq. (4.16). Unlike the case of the Brieskorn homology sphere, we have seen that the exact asymptotic expansion of the WRT invariant for the Seifert manifold with 4-singular fibers does not have a form of eq. (4.16) rather eq. (4.5) as was pointed out in Ref. 24. Especially the number of the dominating exponential terms is not proportional to the Casson invariant, and all the irreducible SU(2) representation of the fundamental group (1.4) do not appear as we can read off from eq. (4.14).

The representation space of the fundamental group $\pi_1(\mathcal{M})$ of the Seifert manifold $\mathcal{M} = \Sigma(p_1,p_2,p_3,p_4)$ with 4-singular fibers was investigated in detail in Refs. 3, 13 (see also Ref. 29). There are two types of the irreducible representation of the fundamental group (1.4), $\rho : \pi_1\big(\Sigma(p_1,p_2,p_3,p_4)\big) \to SU(2)$, up to conjugation;

- one of the generators $x_k$ is mapped to $\pm\,\mathrm{id}$,
- all images $\rho(x_k)$ differ from $\pm\,\mathrm{id}$.

As the former case can be given from the representation space of the Brieskorn homology sphere, eq. (4.14) shows that the number of lattice points $\gamma(p_1,p_2,p_3,p_4)$ is related to the number of the latter case. See Section 5 for some examples. We can thus conclude that the latter type of the irreducible representations of $\pi_1(\mathcal{M})$ dominates the asymptotic behavior of the WRT invariant in $N \to \infty$.

One may expect that the "missing" irreducible representations of $\pi_1(\mathcal{M})$, in which one of generators $x_k$ is mapped to $\pm\,\mathrm{id}$, correspond to the next leading terms $Z^{(1)}_{N-2}(\mathcal{M})$ (4.8) or (4.9). Actually they give the non-zero contribution, but we can see that the number of non-zero terms in $Z^{(1)}_{N-2}(\mathcal{M})$ is not equal to that of missing representations.

Once we have seen that the dominating exponential factor of the WRT invariant can be interpreted from the SU(2) irreducible representation of the fundamental group $\pi_1(\mathcal{M})$, we can find that the asymptotic behavior gives the Chern–Simons invariant of the manifold. Explicitly the Chern–Simons invariant for the Seifert manifold is written from the exponential factor of eq. (4.5) as

$$\mathrm{CS}(A) = -\frac{P}{4}\left(1+\sum_{j=1}^{4}\frac{\ell_j}{p_j}\right)^2 \mod 1 \tag{4.17}$$

which originally appears as the phase of the $\mathbf{T}$-matrix (2.17) of the modular form $\Phi^{\vec{\ell}}_{\vec{p}}(\tau)$.

## 4.2 Contribution from Trivial Connections

In asymptotic behavior of the WRT invariant $\tau_N(\mathcal{M})$ in $N \to \infty$, the exponential terms are dominating, but a tail part has its own meaning as a contribution from trivial connections (see

Refs. 16, 24). We further show that this corresponds to the Ohtsuki invariant for the Seifert manifold.

Before discussing a connection with the quantum invariant, we give a generating function of the $T$-series in eqs. (4.2) and (4.4).

**Proposition 11.** *Let the $T$-series $T_{\lesseqgtr}(k)$ be defined by eqs.* (4.2) *and* (4.4)*. We have*

$$\frac{\prod_{j=1}^{4}\sinh\left(\frac{P}{p_j}\,x\right)}{\bigl(\sinh(P\,x)\bigr)^2}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{T_{\lesseqgtr}(k)}{(2\,k)!}\,x^{2k} \tag{4.18}$$

*depending on* $\sum_{j=1}^{4}\frac{1}{p_j}\lesseqgtr 1$.

*Proof.* We first study a case of $\sum_j\frac{1}{p_j}<1$. We have from eq. (3.7)

$$\begin{aligned}\frac{\prod_{j=1}^{4}\left(z^{P/p_j}-z^{-P/p_j}\right)}{\left(z^P-z^{-P}\right)^2}&=-\sum_{n=0}^{\infty}\left(1+\left\lfloor\frac{n}{2\,P}\right\rfloor\right)\chi_{2P}^{(1,1,1,1)}(n)\,z^{n+P}\\&=\sum_{k=0}^{\infty}\sum_{n=P}^{3P-1}(1+k)\,\chi_{2P}^{(p_1-1,1,1,1)}(n)\,z^{2Pk+n}\end{aligned}$$

We substitute $z=\mathrm{e}^{-x}$ for the above expression, and we equate it with $\sum_{k=0}^{\infty}T_k\,x^{2k}$ in a limit $x\searrow 0$. Applying the Mellin transformation, we get

$$T_k=\frac{(2\,P)^{2k}}{(2\,k)!}\sum_{n=P}^{3P-1}\chi_{2P}^{(p_1-1,1,1,1)}(n)\left(\zeta\left(-2\,k-1,\frac{n}{2\,P}\right)-\frac{n-2\,P}{2\,P}\,\zeta\left(-2\,k,\frac{n}{2\,P}\right)\right) \tag{4.19}$$

Using an analytic continuation of the Hurwitz zeta function

$$\zeta(1-k,z)=-\frac{B_k(z)}{k}$$

for $k\in\mathbb{Z}_{>0}$, we get the statement of the theorem.

In a case of $\sum_j\frac{1}{p_j}>1$, we recall eq. (3.11) which gives

$$\frac{\prod_{j=1}^{4}\left(z^{P/p_j}-z^{-P/p_j}\right)}{\left(z^P-z^{-P}\right)^2}=\sum_{k=0}^{\infty}\sum_{n=P}^{3P-1}(1+k)\,\chi_{2P}^{(p_1-1,1,1,1)}(n)\,z^{2Pk+n}+\sum_{k=0}^{\infty}\psi_{2P}^{(2P-P\sum_j\frac{1}{p_j})}(k)\,z^k$$

By the Mellin transformation after setting $z=\mathrm{e}^{-x}$, the first term of RHS gives eq. (4.19) while the second term gives $\frac{1}{(2k)!}\,L(-2\,k,\psi_{2P}^{(2P-P\sum_j\frac{1}{p_j})})$ as a coefficient of $x^{2k}$. Combining these results, we obtain eq. (4.18). □

We note that some of the $T$-series are explicitly computed as follows;

$$
\begin{aligned}
T_{\lessgtr}(0) &= 0 \\
T_{\lessgtr}(1) &= 4\,P \\
T_{\lessgtr}(2) &= 8\,P^3 \left( -2 + \sum_{j=1}^{4} \frac{1}{{p_j}^2} \right) \\
T_{\lessgtr}(3) &= 4\,P^5 \left( 5 \left( 2 - \sum_{j=1}^{4} \frac{1}{{p_j}^2} \right)^2 + 2 \left( 2 - \sum_{j=1}^{4} \frac{1}{{p_j}^4} \right) \right)
\end{aligned}
$$

The fact that coefficients in tail part $T_{\lessgtr}(k)$ of the asymptotic expansion of the WRT invariant also appear in the Taylor series (4.18) was pointed out in Ref. 16 by the different manner. It is noted that we also have a similar connection between the Eichler integral and the colored Jones polynomial for torus knots/links, and that the inverse of the Alexander polynomial generates this tail part of the $N$-colored Jones polynomial at the $N$-th root of unity.

See that the generating function (4.18) of $T_{\lessgtr}(k)$ has already appeared in the WRT invariant (2.2) of the Seifert manifold. The $T$-series is still related to the quantum invariant, *i.e.*, the Ohtsuki invariant. The $n$-th Ohtsuki invariant $\lambda_n(\mathcal{M})$ for 3-manifold $\mathcal{M}$ is defined by [19, 21]

$$
\tau_\infty(\mathcal{M}) = \sum_{n=0}^{\infty} \lambda_n(\mathcal{M})\,(q-1)^n \tag{4.20}
$$

where the formal power series $\tau_\infty(\mathcal{M})$ is defined from a tail part of the asymptotic expansion of the WRT invariant, and in our case of the Seifert manifold $\mathcal{M} = \Sigma(p_1, p_2, p_3, p_4)$ with 4-singular fibers we have from Theorem 6 that

$$
q^{\frac{\phi}{4} - \frac{1}{2}}\,(q-1) \cdot \tau_\infty\big(\Sigma(p_1, p_2, p_3, p_4)\big) = \sum_{k=0}^{\infty} \frac{T_{\lessgtr}(k)}{k!} \left( \frac{\log q}{4\,P} \right)^k \tag{4.21}
$$

if we replace the $N$-th root of unity with a parameter $q$, $q \leftrightarrow \exp\left(\frac{2\pi\mathrm{i}}{N}\right)$. Using the Stirling number of the first kind $S_n^{(m)}$ defined by

$$
\prod_{j=0}^{n-1} (x-j) = \sum_{m=0}^{n} S_n^{(m)}\,x^m
$$

which satisfies (see *e.g.* Ref. 1)

$$
\frac{(\log q)^m}{m!} = \sum_{n=m}^{\infty} S_n^{(m)}\,\frac{(q-1)^n}{n!}
$$

we obtain the following formula;

**Proposition 12.** *The $n$-th Ohtsuki invariant $\lambda_n(\mathcal{M})$ for the Seifert manifold $\mathcal{M} = \Sigma(p_1,p_2,p_3,p_4)$ is given by*

$$\lambda_n\bigl(\Sigma(p_1,p_2,p_3,p_4)\bigr) = \sum_{j=0}^{n} \binom{\frac{2-\phi}{4}}{n-j} \sum_{k=1}^{j+1} \frac{T_{\lessgtr}(k)}{(4\,P)^k} \, \frac{S_{j+1}^{(k)}}{(j+1)!} \tag{4.22}$$

One can check that the first three terms are computed explicitly as follows;

$$\begin{aligned}
\lambda_0 &= 1 \\
\lambda_1 &= 6\,\lambda_C\bigl(\Sigma(p_1,p_2,p_3,p_4)\bigr) \\
\lambda_2 &= \frac{3\,\phi^2 + 12\,\phi + 4}{96} + \frac{P}{16}\,(\phi+2)\left(2 - \sum_j \frac{1}{{p_j}^2}\right) \\
&\qquad + \frac{P^2}{96}\left(2\left(2 - \sum_j \frac{1}{{p_j}^4}\right) + 5\left(2 - \sum_j \frac{1}{{p_j}^2}\right)^2\right)
\end{aligned}$$

The fact that $\lambda_1$ is equal to 6 times of the Casson invariant was first pointed out in Ref. 19, and $\lambda_2$ was derived in Ref. 28. See also Ref. 15.

## 5. Examples

### 5.1 $\Sigma(2,3,5,7)$

With $\vec{p} = (2,3,5,7)$ the modular form spans $D = 6$ dimensional space, and the independent periodic functions $\chi_{420}^{\vec{\ell}}(n)$ are defined when quadruples $\vec{\ell}$ are $(1,1,1,1)$, $(1,1,1,2)$, $(1,1,1,3)$, $(1,1,2,1)$, $(1,1,2,2)$, and $(1,1,2,3)$. We have $\sum_j \frac{1}{p_j} = \frac{247}{210} > 1$, and we can check $C_{\vec{p}}(\vec{\ell}) \neq 0$ for all $\vec{\ell}$. Indeed from eq. (4.12) we have $\gamma(2,3,5,7) = 6$. These quadruples correspond to the irreducible SU(2) representation of the fundamental group (1.4), in which none of the generators $x_k$ is mapped to $\pm\,\mathrm{id}$. The Casson invariant is computed as $\lambda_C\bigl(\Sigma(2,3,5,7)\bigr) = -14$, and we have missing irreducible representations in which one of generators is mapped to $\pm\,\mathrm{id}$.

See Tables 1 and 2, which should be compared with a table in Ref. 3. In Table 1, collected are quadruples $\vec{\ell}$, which contribute to the asymptotic behavior (4.5) of the WRT invariant. These denote the SU(2) representations of $\pi_1(\mathcal{M})$, which do not map any generators $x_k$ (1.4) to $\pm\,\mathrm{id}$. We have listed quadruples $\vec{\ell}$ in Table 2, and they correspond to the irreducible representations missing in Table 1. The number of these missing representations is proportional to the sum of the Casson invariant, $\lambda_C\bigl(\Sigma(2,3,5)\bigr) + \lambda_C\bigl(\Sigma(2,3,7)\bigr) + \lambda_C\bigl(\Sigma(2,5,7)\bigr) + \lambda_C\bigl(\Sigma(3,5,7)\bigr)$. As the representation in Table 1 have 2-dimensional components, we indeed recover the Casson invariant by $-\frac{1}{2}\,(2\times 6 + (2+2+4+8)) = -14$.

| $\vec{\ell}$ | $\sum_{j=1}^{4} \frac{\ell_j}{p_j}$ | $C_{\vec{p}}(\vec{\ell})$ | $\mathrm{CS}(\mathcal{M})$ |
|---|---|---|---|
| $(1,1,1,1)$ | $\frac{247}{210}$ | $\frac{37}{105}$ | $-\frac{529}{840}$ |
| $(1,1,1,2)$ | $\frac{277}{210}$ | $\frac{67}{105}$ | $-\frac{289}{840}$ |
| $(1,1,1,3)$ | $\frac{307}{210}$ | $\frac{4}{5}$ | $-\frac{169}{840}$ |
| $(1,1,2,1)$ | $\frac{289}{210}$ | $\frac{4}{7}$ | $-\frac{361}{840}$ |
| $(1,1,2,2)$ | $\frac{319}{210}$ | $\frac{109}{105}$ | $-\frac{121}{840}$ |
| $(1,1,2,3)$ | $\frac{349}{210}$ | $\frac{139}{105}$ | $-\frac{1}{840}$ |

Table 1: $\mathcal{M} = \Sigma(2,3,5,7)$: Listed are quadruples $\vec{\ell}$, which contribute to the asymptotics of the WRT invariant $Z_N(\mathcal{M})$. These quadruples denote the irreducible representations of $\pi_1(\mathcal{M})$ (1.4), in which none of the generators $x_k$ is mapped to $\pm\,\mathrm{id}$. The Chern–Simons invariant $\mathrm{CS}(\mathcal{M})$ is computed from eq. (4.17).

| $\vec{\ell}$ | $\sum_j \frac{\ell_j}{p_j}$ | $\mathrm{CS}(\mathcal{M})$ |
|---|---|---|
| $(1,1,1,0)$ | $\frac{31}{30}$ | $-\frac{7}{120}$ |
| $(1,1,2,0)$ | $\frac{37}{30}$ | $-\frac{103}{120}$ |
| $(1,1,0,2)$ | $\frac{47}{42}$ | $-\frac{125}{168}$ |
| $(1,1,0,3)$ | $\frac{53}{42}$ | $-\frac{101}{168}$ |
| $(1,0,1,3)$ | $\frac{79}{70}$ | $-\frac{243}{280}$ |
| $(1,0,2,1)$ | $\frac{73}{70}$ | $-\frac{27}{280}$ |
| $(1,0,2,2)$ | $\frac{83}{70}$ | $-\frac{227}{280}$ |
| $(1,0,2,3)$ | $\frac{93}{70}$ | $-\frac{187}{280}$ |
| $(0,1,1,4)$ | $\frac{116}{105}$ | $-\frac{121}{210}$ |
| $(0,1,1,5)$ | $\frac{131}{105}$ | $-\frac{23}{105}$ |
| $(0,1,1,6)$ | $\frac{146}{105}$ | $-\frac{1}{210}$ |
| $(0,1,2,2)$ | $\frac{107}{105}$ | $-\frac{2}{105}$ |
| $(0,1,2,3)$ | $\frac{122}{105}$ | $-\frac{79}{210}$ |
| $(0,1,2,4)$ | $\frac{137}{105}$ | $-\frac{92}{105}$ |
| $(0,1,2,5)$ | $\frac{152}{105}$ | $-\frac{109}{210}$ |
| $(0,1,2,6)$ | $\frac{167}{105}$ | $-\frac{32}{105}$ |

Table 2: $\mathcal{M} = \Sigma(2,3,5,7)$: Listed are quadruples, which correspond to the representations of the fundamental group $\pi_1(\mathcal{M})$ in which one of generators $x_k$ is mapped to $\pm\,\mathrm{id}$.

In Table 3 we give a result of numerical computations, and compare with the exact value (2.2) of the WRT invariant with the asymptotic formula (4.5). We have also given numerical values (4.6) including the next leading terms $Z_{N-2}^{(1)}(\mathcal{M})$ to find a good agreement. All these computations are performed on PARI/GP. We stress that the representations $\vec{\ell}$ in Table 1 dominate the asymptotic behavior of the WRT invariant in $N \to \infty$.

### 5.2 $\Sigma(3,4,5,7)$

We have $\sum_j 1/p_j = \frac{389}{420} < 1$, and eq. (2.12) gives $D = 18$. We find that $\gamma(3,4,5,7) = 17$ from eq. (4.12). Indeed among 18 independent quadruples $\vec{\ell}$, we see that $C_{3,4,5,7}(1,1,1,1) = 0$. See Table 4 for representations $\vec{\ell}$, in which none of generators $x_k$ of the fundamental group (1.4) is mapped to $\pm\,\mathrm{id}$. The Casson invariant is computed as $\lambda_C\big(\Sigma(3,4,5,7)\big) = -31$, and we have missing representations. As the representations in Table 4 have 2-dimensional components, missing representations can be given for the Seifert manifold with 3-singular fibers, and its number is proportional to the sum of the Casson invariant, $\lambda_C\big(\Sigma(3,4,5)\big) = -2$, $\lambda_C\big(\Sigma(3,4,7)\big) = -3$, $\lambda_C\big(\Sigma(3,5,7)\big) = -4$, and $\lambda_C\big(\Sigma(4,5,7)\big) = -5$.

In Table 5, we compute numerically both the exact value (2.2) and the asymptotic value (4.5) of the WRT invariant. We can see a good agreement also in this case.

## 6. DISCUSSIONS

We have studied the asymptotic expansion of the WRT invariant for the Seifert manifold with 4-singular fibers. A key is that the WRT invariant can be rewritten in terms of a limiting value of the Eichler integrals of the modular forms with half-integral weight. A close connection between the quantum invariant and the modular form was first observed in Ref. 17 for a case of the Poincaré homology sphere.

In the case of 3-singular fibers, we showed in our previous paper [7] that the number of the non-vanishing Eichler integral coincides with the number of the integral lattice points inside the 3-dimensional tetrahedron. As in known from Ref. 20, the number of the irreducible representations of the fundamental group is related to that of the lattice points, and this number is proportional to the Casson invariant. In this paper, as a generalization to the case of 4-singular fibers, we have shown that the number of the lattice points in the 4-dimensional simplex is related to the number of the non-vanishing Eichler integrals. We have also clarified a relationship with the representation of the fundamental group.

Our modular form with half-integral weight can be easily generalized as follows. We fix $M$-tuple $\vec{p} = (p_1, p_2, \dots, p_M)$, where $p_i \geq 2$ are pairwise coprime positive integers. We set

| $N$ | exact result for $Z_N$ | asymptotic formula $\begin{cases}(N+2)\cdot Z_N^{(0)}+Z_N^{(1)}\\(N+2)\cdot Z_N^{(0)}\end{cases}$ |
|---|---|---|
| 10 | $0.739637+2.732051\,\mathrm{i}$ | $0.740798+2.732420\,\mathrm{i}$ |
| | | $1.259843+2.437384\,\mathrm{i}$ |
| 11 | $0.979154-0.903934\,\mathrm{i}$ | $0.980292-0.903502\,\mathrm{i}$ |
| | | $0.879253-0.968693\,\mathrm{i}$ |
| 12 | $-0.01437482+0.0\,\mathrm{i}$ | $-0.01326139+0.00048566\,\mathrm{i}$ |
| | | $-0.00073383+0.00867459\,\mathrm{i}$ |
| 13 | $-0.04815996-0.13669\,\mathrm{i}$ | $-0.04707020-0.136157687\,\mathrm{i}$ |
| | | $-0.00575862+0.007899715\,\mathrm{i}$ |
| 14 | $-1.864025-0.7606090\,\mathrm{i}$ | $-1.862959-0.7600360\,\mathrm{i}$ |
| | | $-1.995763-1.0444076\,\mathrm{i}$ |
| 100 | $-4.8885515+17.7770857\,\mathrm{i}$ | $-4.8884355+17.7779834\,\mathrm{i}$ |
| | | $-4.8802978+18.1353532\,\mathrm{i}$ |
| 101 | $22.22565+7.134420\,\mathrm{i}$ | $22.22576+7.135315\,\mathrm{i}$ |
| | | $22.39565+7.174075\,\mathrm{i}$ |
| 102 | $-0.6760584-6.611630\,\mathrm{i}$ | $-0.6759530-6.610738\,\mathrm{i}$ |
| | | $-0.6140020-6.398063\,\mathrm{i}$ |
| 103 | $0.28253575+0.2563299\,\mathrm{i}$ | $0.28263601+0.25721904\,\mathrm{i}$ |
| | | $0.00417075+0.06066500\,\mathrm{i}$ |
| 104 | $-0.1814129-6.39840405\,\mathrm{i}$ | $-0.1813177-6.39751779\,\mathrm{i}$ |
| | | $-0.2590138-6.54593985\,\mathrm{i}$ |
| 1000 | $-86.3814448-52.1955841\,\mathrm{i}$ | $-86.3815556-52.1955281\,\mathrm{i}$ |
| | | $-86.3595930-52.0529208\,\mathrm{i}$ |
| 1001 | $-32.1688750+226.931025\,\mathrm{i}$ | $-32.1689857+226.931025\,\mathrm{i}$ |
| | | $-32.0565661+227.078420\,\mathrm{i}$ |
| 1002 | $112.342695+21.8373199\,\mathrm{i}$ | $112.342584+21.8373757\,\mathrm{i}$ |
| | | $112.122297+21.6486285\,\mathrm{i}$ |
| 1003 | $0.7904096329+0.9244664554\,\mathrm{i}$ | $0.7902992554+0.9245221724\,\mathrm{i}$ |
| | | $0.6333213866+1.0639142783\,\mathrm{i}$ |
| 1004 | $57.8951433+129.671829\,\mathrm{i}$ | $57.8950330+129.671885\,\mathrm{i}$ |
| | | $58.1582795+130.079477\,\mathrm{i}$ |
| 1005 | $69.6412229-74.3079505\,\mathrm{i}$ | $69.6411128-74.3078950\,\mathrm{i}$ |
| | | $69.5928674-74.2038809\,\mathrm{i}$ |
| 10000 | $527.74686902+862.13517540\,\mathrm{i}$ | $527.74686459+862.13517563\,\mathrm{i}$ |
| | | $527.55843581+861.79480366\,\mathrm{i}$ |
| 10001 | $-6.393230664+1.730198170\,\mathrm{i}$ | $-6.393235091+1.730198403\,\mathrm{i}$ |
| | | $-6.624993671+1.763443057\,\mathrm{i}$ |
| 10002 | $-301.5164629-551.4353628\,\mathrm{i}$ | $-301.5164673-551.4353625\,\mathrm{i}$ |
| | | $-301.7624718-551.3640027\,\mathrm{i}$ |
| 10003 | $-10.84155396-1.9919898321\,\mathrm{i}$ | $-10.84155838-1.9919895988\,\mathrm{i}$ |
| | | $-10.95621827-1.9107737795\,\mathrm{i}$ |
| 10004 | $-868.9478695-736.82135025\,\mathrm{i}$ | $-868.9478739-736.8213500\,\mathrm{i}$ |
| | | $-869.2651992-736.9897060\,\mathrm{i}$ |

Table 3: Numerical values of the WRT invariant $Z_N(\mathcal{M})$ for $\mathcal{M}=\Sigma(2,3,5,7)$. We have used $P=210$ and $\phi=\frac{949}{210}$ in eq. (2.2).

| $\vec{\ell}$ | $\sum_j \frac{\ell_j}{p_j}$ | $C_{\vec{p}}(\vec{\ell})$ | $\mathrm{CS}(\mathcal{M})$ |
|---|---|---|---|
| $(1,1,1,2)$ | $\frac{449}{420}$ | $\frac{29}{210}$ | $-\frac{841}{1680}$ |
| $(1,1,1,3)$ | $\frac{509}{420}$ | $\frac{89}{210}$ | $-\frac{1201}{1680}$ |
| $(1,1,1,4)$ | $\frac{569}{420}$ | $\frac{149}{210}$ | $-\frac{361}{1680}$ |
| $(1,1,1,5)$ | $\frac{629}{420}$ | $\frac{4}{5}$ | $-\frac{1}{1680}$ |
| $(1,1,1,6)$ | $\frac{689}{420}$ | $\frac{109}{210}$ | $-\frac{121}{1680}$ |
| $(1,1,2,1)$ | $\frac{473}{420}$ | $\frac{53}{210}$ | $-\frac{1129}{1680}$ |
| $(1,1,2,2)$ | $\frac{533}{420}$ | $\frac{113}{210}$ | $-\frac{1009}{1680}$ |
| $(1,1,2,3)$ | $\frac{593}{420}$ | $\frac{173}{210}$ | $-\frac{1369}{1680}$ |
| $(1,1,2,4)$ | $\frac{653}{420}$ | $1$ | $-\frac{529}{1680}$ |
| $(1,1,2,5)$ | $\frac{713}{420}$ | $\frac{197}{210}$ | $-\frac{169}{1680}$ |
| $(1,1,2,6)$ | $\frac{773}{420}$ | $\frac{4}{7}$ | $-\frac{289}{1680}$ |
| $(1,2,1,1)$ | $\frac{247}{210}$ | $\frac{37}{105}$ | $-\frac{109}{420}$ |
| $(1,2,1,2)$ | $\frac{277}{210}$ | $\frac{67}{105}$ | $-\frac{289}{420}$ |
| $(1,2,1,3)$ | $\frac{307}{210}$ | $\frac{4}{5}$ | $-\frac{169}{420}$ |
| $(1,2,2,1)$ | $\frac{289}{210}$ | $\frac{4}{7}$ | $-\frac{361}{420}$ |
| $(1,2,2,2)$ | $\frac{319}{210}$ | $\frac{109}{105}$ | $-\frac{121}{420}$ |
| $(1,2,2,3)$ | $\frac{349}{210}$ | $\frac{139}{105}$ | $-\frac{1}{420}$ |

Table 4: $\mathcal{M} = \Sigma(3,4,5,7)$: We give the SU(2) irreducible representations of the fundamental group which contribute to the asymptotics of the WRT invariant $Z_N(\mathcal{M})$. None of the generators of the fundamental group (1.4) is mapped to $\pm\,\mathrm{id}$.

$P = \prod_{j=1}^M p_j$, and define the periodic function by

$$\chi_{2P}^{\vec{\ell}}(n) = \begin{cases} -\displaystyle\prod_{j=1}^M \varepsilon_j & \text{if } n \equiv P\left(1 + \displaystyle\sum_{j=1}^M \varepsilon_j \frac{\ell_j}{p_j}\right) \mod 2\,P \\ 0 & \text{others} \end{cases} \tag{6.1}$$

where $\varepsilon_j \in \{1,-1\}$, and $\vec{\ell} = (\ell_1, \dots, \ell_M) \in \mathbb{Z}^M$ is an $M$-tuple satisfying $0 < \ell_j < p_j$. The case of $M = 2$ appears as the Virasoro character of the minimal model. There are $2^M$ non-zero terms $\chi_{2P}^{\vec{\ell}}(n)$ in one period $n \in [0, 2\,P]$, and we have a mean value zero

$$\sum_{n=0}^{2\,P-1} \chi_{2P}^{\vec{\ell}}(n) = 0$$

| $N$ | exact result for $Z_N$ | asymptotic formula $\begin{cases}(N+2)\cdot Z_N^{(0)}+Z_N^{(1)}\\(N+2)\cdot Z_N^{(0)}\end{cases}$ |
|---|---|---|
| 998 | $170.573359-7.19243844\,\mathrm{i}$ | $170.573296-7.19236552\,\mathrm{i}$ |
| | | $170.574879-7.19150956\,\mathrm{i}$ |
| 999 | $1.981255663-0.539792723\,\mathrm{i}$ | $1.981192840-0.539719917\,\mathrm{i}$ |
| | | $2.018358388-0.430004395\,\mathrm{i}$ |
| 1000 | $10.9510287+54.4329988\,\mathrm{i}$ | $10.9509659+54.4330715\,\mathrm{i}$ |
| | | $10.9917059+54.5856899\,\mathrm{i}$ |
| 1001 | $124.831004-123.107089\,\mathrm{i}$ | $124.830941-123.107017\,\mathrm{i}$ |
| | | $124.881006-123.189839\,\mathrm{i}$ |
| 1002 | $61.4397540-97.5216937\,\mathrm{i}$ | $61.4396912-97.5216212\,\mathrm{i}$ |
| | | $61.2653019-97.4592987\,\mathrm{i}$ |
| 1003 | $146.275643+49.1637851\,\mathrm{i}$ | $146.275580+49.1638575\,\mathrm{i}$ |
| | | $146.394939+49.4235715\,\mathrm{i}$ |
| 1004 | $172.965961+71.0176797\,\mathrm{i}$ | $172.965898+71.0177519\,\mathrm{i}$ |
| | | $173.332924+71.1299446\,\mathrm{i}$ |
| 1005 | $99.17707586-1.446353766\,\mathrm{i}$ | $99.17701318-1.446281609\,\mathrm{i}$ |
| | | $99.07532715-1.557181709\,\mathrm{i}$ |
| 1006 | $3.371067321-1.677974019\,\mathrm{i}$ | $3.371004661-1.677901969\,\mathrm{i}$ |
| | | $3.385944423-1.713271357\,\mathrm{i}$ |
| 1007 | $67.04084613-35.56880740\,\mathrm{i}$ | $67.04078349-35.56873545\,\mathrm{i}$ |
| | | $67.08289485-35.47827521\,\mathrm{i}$ |

Table 5: Numerical values of the WRT invariant $Z_N(\mathcal{M})$ for $\mathcal{M}=\Sigma(3,4,5,7)$. We have used $P=420$ and $\phi=\frac{961}{420}$ in eq. (2.2).

One sees that the function $\chi_{2P}^{\vec{\ell}}(n)$ is even (resp. odd) if $M$ is even (resp. odd). When we define the functions $\Phi_{\vec{p}}^{\vec{\ell}}(\tau)$ by

$$\Phi_{\vec{p}}^{\vec{\ell}}(\tau)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{Z}}\chi_{2P}^{\vec{\ell}}(n)\,q^{\frac{n^2}{4P}} & \text{if } M \text{ is even}\\ \dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{Z}}n\,\chi_{2P}^{\vec{\ell}}(n)\,q^{\frac{n^2}{4P}} & \text{if } M \text{ is odd}\end{cases} \tag{6.2}$$

we find that it is a vector modular form with dimension

$$D=\frac{1}{2^{M-1}}\prod_{j=1}^{M}(p_j-1)$$

due to the symmetry under the involutions $\sigma_{i,j}$, and that the weight is $1/2$ (resp. $3/2$) when $M$ is even (resp. odd). Generally the SU(2) WRT invariant for the Seifert manifold $\Sigma(p_1, \dots, p_M)$ with $M$-singular fibers is related to this vector modular form.

As a generalization of the correspondence between the lattice points and the non-vanishing Eichler integrals, we propose the following conjecture.

**Conjecture 1.** *If we set $\gamma(\vec{p})$ as the number of $M$-tuples $\vec{\ell}$ satisfying*

$$\sum_{n=1}^{2P} \chi_{2P}^{\vec{\ell}}(n) \, B_{M-2}\left(\frac{n}{2P}\right) \neq 0$$

*then $D - \gamma(\vec{p})$ coincides with the number of the lattice points satisfying*

$$0 < \sum_{j=1}^{M} \frac{\ell_j}{p_j} < 1$$

A case of $M = 2$ is trivial (see Ref. 10), as we have $\gamma = 0$ due to $B_0(x) = 1$ and the definition of $\sum_{n=1}^{2P} \chi_{2P}^{\vec{\ell}}(n) = 0$. A case of $M = 3$ was shown in Ref. 7, and we have proved this conjecture for $M = 4$ in this paper.

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